UNIVERSITY OF PERUGIA

JOB CREATION ORIENTED

1st LEVEL DEGREE IN BIOTECNOLOGIES

A.A. 2004-05

Module:

ELEMENTS OF PHYSICS

AND OF DYNAMICS OF FLUIDS

Prof. Renzo Campanella

Table of contents

Program of the module

Introduction
Physical quantities – Dimensions and units – Idealization, Approximation – Errors

Vectors
Frames of reference – Components – Operations on vectors: addition and subtraction of vectors, multiplication of a vector times a scalar

Kinematics
Motion in a straight line – Average and instantaneous velocity – Average and instantaneous acceleration – The equation of uniformly accelerated motion – Two dimensional motion – Uniform circular motion and its components – Harmonic motion

Dinamics
Newton's laws of motion – Weight – Normal force – Friction –Drag – Centrifugal force – Elastic force – Momentum – Conservation of momentum – Examples of conservation of momentum: cannon and bullet, man on the boat – Example of motion in presence of drag: Immunoglobulin

Examples of dynamics
The centrifuge

Work and energy
Kinetic energy – Work – Theorem of kinetic energy – Potential energy – Conservative forces – Conservation of mechanical energy – Work of non-conservative forces – Energy of the harmonic oscillator

Angular quantities
Angular velocity – Angular acceleration – Moment of inertia – Rotational kinetic energy – Angular momentum – Torque

Fluids
The Nature of Fluids – Pressure – Hydrostatic pressure, buoyancy – Archimede's principle – The equations of statics of fluids – Stevino's law – Ideal fluids – The continuity equation – Bernoulli's equation – Torricelli's law – The Venturi's tube – Viscosity and the flow of real fluids – Velocity profile for a viscous fluid – Poiseuille's equation

Mail: campanella@unipg.it
Website: http://www.unipg.it/campanella.

In the website, go to the English page.
E-help: in the forum you may post any question, which will be answered under the same thread. Replies from other students are welcome.
In the case that a meeting is needed, send an email.

Chapter 1: Vectors

VECTORS

Frames of reference, components

Figure 1.1 shows a cartesian frame of reference, formed by three mutually orthogonal axes *x*, *y* and *z*, with a common point O called origin. On each axis a positive orientation is defined by means of an arrow. Along the three axes three segments $\hat{i}$, $\hat{j}$ and $\hat{k}$, whose length is conventionally chosen equal to one define the directions of the three axes. $\hat{i}$, $\hat{j}$ and $\hat{k}$ are called versors.

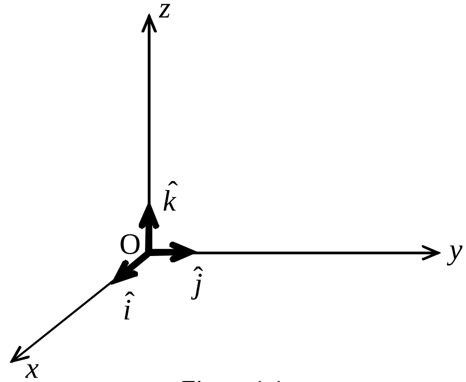


Figure 1.1

Figure 1.2 shows a vector, i.e. an oriented segment defined by two points P and Q. A vector is denoted with a symbol $\vec{v}$; P' and Q' are the projection of P and Q on the *xy* plane, i.e. the points at the intersections of the *xy* plane with a perpendicular line drawn from the two points. From P' and Q' we obtain the projections x_P, x_Q and y_P, y_Q simply by tracing the lines parallel (respectively) to the *y* and *x* axes. The segments $(x_Q - x_P)$, $(y_Q - y_P)$ and $(z_Q - z_P)$ are called the components of PQ along *x*, *y* and *z*. As can be seen from the figure 1.2, in our example $(x_Q - x_P)$ is negative, being x_Q smaller than x_P, whilst $(y_Q - y_P)$ and $(z_Q - z_P)$ are positive.

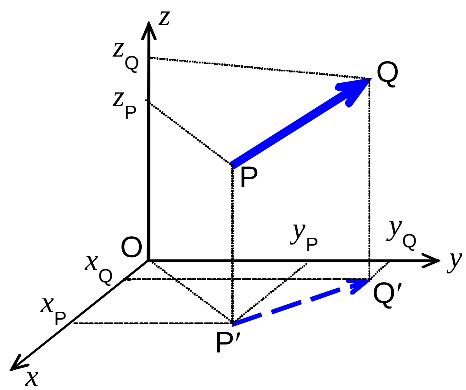


Figure 1.2

Given the components $(x_Q - x_P)$, $(y_Q - y_P)$ and $(z_Q - z_P)$, the vector is:

$$\vec{v} = \left(x_Q - x_P\right)\hat{i} + \left(y_Q - y_P\right)\hat{j} + \left(z_Q - z_P\right)\hat{k} \tag{1.1}$$

and we shall usually write:

$$\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k} \tag{1.2}$$

where, obviously

$$\begin{cases} v_x = \left(x_Q - x_P\right) \\ v_y = \left(y_Q - y_P\right) \\ v_z = \left(z_Q - z_P\right) \end{cases}$$

Sum and difference of vectors, product of a vector times a scalar

The sum of two vectors $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ and $\vec{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$ is defined as $\vec{c} = \vec{a} + \vec{b}$. By the definition (1.2) we have:

$$\vec{c} = \left(a_x\hat{i} + a_y\hat{j} + a_z\hat{k}\right) + \left(b_x\hat{i} + b_y\hat{j} + b_z\hat{k}\right) = \left(a_x + b_x\right)\hat{i} + \left(a_y + b_y\right)\hat{j} + \left(a_z + b_z\right)\hat{k} = c_x\hat{i} + c_y\hat{j} + c_z\hat{k}$$

with

$$\begin{cases} c_x = a_x + b_x \\ c_y = a_y + b_y \\ c_z = a_z + b_z \end{cases}$$

A graphical depiction of the sum of two vectors is in figure 1.3. The vector $\vec{c}$ connects the extremities of the two vectors $\vec{a}$ and $\vec{b}$.

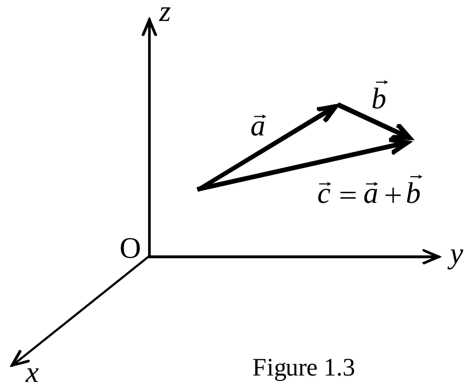


Figure 1.3

The difference of two vectors $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ and $\vec{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$ is defined as $\vec{c} = \vec{a} - \vec{b}$. By the definition (1.2) we have:

$$\vec{c} = \left(a_x\hat{i} + a_y\hat{j} + a_z\hat{k}\right) - \left(b_x\hat{i} + b_y\hat{j} + b_z\hat{k}\right) = \left(a_x - b_x\right)\hat{i} + \left(a_y - b_y\right)\hat{j} + \left(a_z - b_z\right)\hat{k} = c_x\hat{i} + c_y\hat{j} + c_z\hat{k}$$

with

$$\begin{cases} c_x = a_x - b_x \\ c_y = a_y - b_y \\ c_z = a_z - b_z \end{cases}$$

A graphical depiction of the difference of two vectors is in figure 1.4. The vector $\vec{c} = \vec{a} - \vec{b}$ can be thought of as the vector that added to $\vec{b}$ is equal to $\vec{a}$. The product of a vector times a scalar is the vector $\vec{b}$ defined as $\vec{b} = k\vec{a}$, where k is a scalar. By the definition (1.2) we have:

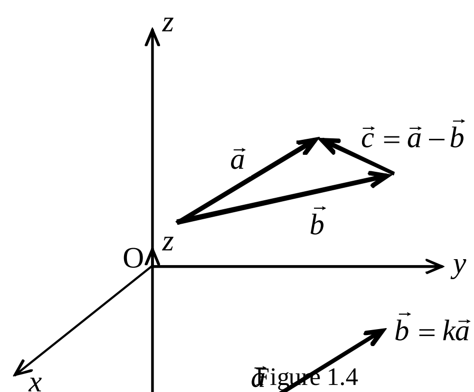


Figure 1.4

$$\vec{b} = k\left(a_x\hat{i} + a_y\hat{j} + a_z\hat{k}\right) = ka_x\hat{i} + ka_y\hat{j} + ka_z\hat{k} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$$

with

$$\begin{cases} b_x = ka_x \\ b_y = ka_y \\ b_z = ka_z \end{cases}$$

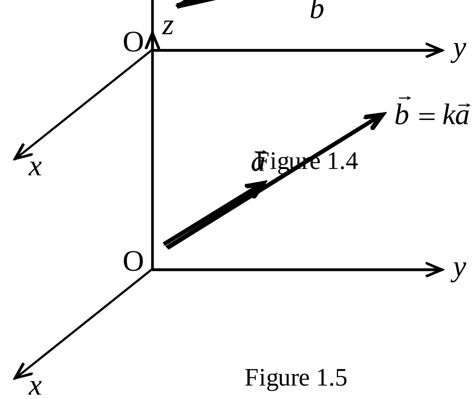


Figure 1.5

A graphical depiction of the product of a vector times a scalar is in figure 1.5.

Chapter 2: Kinematics

MOTION IN ONE DIMENSION

Position, average and instantaneous velocity

Let us imagine a straight oriented line, with an origin O from which the positions on the line are defined as the distanc from O. A point P is moving along the line; be x(t) the position occupied at the instant (t) and x(t+Δt) the position occupied at the instant (t+Δt). We have thus a displacement Δx, given by the difference between the two positions, and a time interval Δt given by the difference between the two instants; they are defined as $\Delta x = x(t+\Delta t) - x(t)$ and $\Delta t = (t+\Delta t) - (t)$. We define as average velocity the quantity

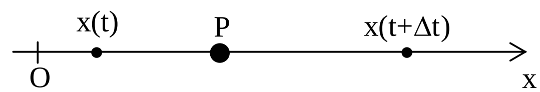


Figure 2.1

$$\overline{v} = \frac{\Delta x}{\Delta t} = \frac{x(t+\Delta t) - x(t)}{\Delta t} \tag{2.1}$$

We recognize the right hand side as the incremental ratio of the variable x with respect to t. The instantaneous velocity is defined as the time derivative of x with respect to t:

$$v(t) = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} = \frac{dx}{dt} \tag{2.2}$$

Average and instantaneous acceleration

The definition of acceleration follows the same scheme: be v(t) the velocity at instant (t) and v(t+Δt) the velocity at the instant (t+Δt). We define the average acceleration

$$\overline{\mathrm{a}} = \frac{v(t+\Delta t) - v(t)}{\Delta t} \tag{2.3}$$

and the instantaneous acceleration

$$\mathrm{a}(\mathrm{t}) = \lim_{\Delta t \to 0} \frac{v(t+\Delta t) - v(t)}{\Delta t} = \frac{dv}{dt} \tag{2.4}$$

The equation of uniformly accelerated motion

In practice the function x(t) is derived starting from (2.4): from $a(t) = \frac{dv}{dt}$ we obtain

$$dv = a(t)\,dt \tag{2.5}$$

which can be integrated:

$$\int dv = \int a(t)\,dt \tag{2.6}$$

If the acceleration a(t) is constant from (2.6) we obtain

$$v(t) = at + C_1 \tag{2.7}$$

From $v(t) = \frac{dx}{dt}$ we obtain:

$$dx = v(t)\,dt \tag{2.8}$$

Inserting (2.7) in (2.8) we obtain:

$$dx = (at + C_1)\,dt \tag{2.9}$$

which can be integrated:

$$\int dx = \int (at + C_1)\,dt = \int atdt + \int C_1 dt = a\int tdt + C_1 \int dt$$

which gives

$$x(t) = \frac{1}{2}at^2 + C_1 t + C_2 \tag{2.10}$$

The two constants C_1 and C_2 depend on the particular constraints of the problem, and can be obtained when two different values of position or velocity are given. As an example let us have the values of x(t) and of v(t) for t=0: these conditions, that in this case are called initial conditions, are written as $x(t=0) = x_0$ and $v(t=0) = v_0$. From (2.7) we obtain $C_1 = v_0$, while from (2.10) we obtain $C_2 = x_0$, so that (2.10) is rewritten:

$$x(t) = \frac{1}{2}at^2 + v_0 t + x_0 \tag{2.11}$$

(2.11) is called "equation of motion", and its knowledge gives the position for any value of t, once that x_0, v_0 and a are known. If a(t) is constant then the motion is called uniformly accelerated.
A motion for which $v(t)=v_0$ is constant is called "uniform", and its equation of motion is:

$$x(t) = v_0 t + x_0 \tag{2.12}$$

TWO- AND THREE-DIMENSIONAL MOTION

Velocity and acceleration in two and three dimensions

In two or three dimensions the position of the point P is given by a vector $\vec{r}$; proceeding as in the 1-dimensional case the position at time

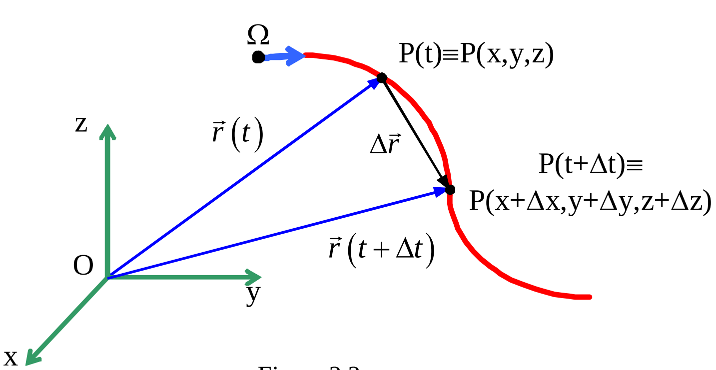


Figure 2.2

(t) is $\vec{r}(t)$ and the position at time (t+Δt) is $\vec{r}(t+\Delta t)$; if we connect the positions defined by the extrimity of the vector $\vec{r}(t)$ for all the values of t we obtain a curve which is called “trajectory”. The positions occupied at the times (t) and (t+Δt) are, respectively, P(t)≡P(x,y,z) and P(t+Δt)≡P(x+Δt,y+Δt,z+Δt). The “displacement” is represented by the vector

$$\Delta\vec{r} = \vec{r}(t+\Delta t) - \vec{r}(t) \tag{2.13}$$

As for the 1-D case we define an average vectorial velocity

$$\bar{\vec{v}} = \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} \tag{2.14}$$

and an instantaneous vectorial velocity

$$\vec{v}(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} = \frac{d\vec{r}}{dt} \tag{2.15}$$

We define also an average vectorial acceleration

$$\bar{\vec{a}} = \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t} \tag{2.16}$$

and an instantaneous vectorial acceleration

$$\vec{a}(t) = \lim_{\Delta t \to 0} \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t} = \frac{d\vec{v}}{dt} \tag{2.17}$$

The vector $\vec{r}(t)$, $\vec{v}(t)$ and $\vec{a}(t)$ are given, in terms of their components [see (1.2)] by the expressions

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k} \tag{2.18}$$

$$\vec{v}(t) = v_x(t)\hat{i} + v_y(t)\hat{j} + v_z(t)\hat{k} \tag{2.19}$$

$$\vec{a}(t) = a_x(t)\hat{i} + a_y(t)\hat{j} + a_z(t)\hat{k} \tag{2.20}$$

For the components the same relations derived in the 1-dimensional case hold, so that in this case we obtain three equations of motion, one for each axis:

$$\begin{cases} x(t) = \frac{1}{2}a_x t^2 + v_{0x}t + x_0 \\ y(t) = \frac{1}{2}a_y t^2 + v_{0y}t + y_0 \\ z(t) = \frac{1}{2}a_z t^2 + v_{0z}t + z_0 \end{cases} \tag{2.21}$$

where (v_{0x}, v_{0y}, v_{0z}) are the components of the initial velocity and (x_0, y_0, z_0) the components of the initial position.

Uniform circular motion and its components

It is a two-dimensional motion characterized by a constant radius R. The point P moves around the circle, and its position can be described by the angle θ or by the arc s. These two quantities are related by the relations

$$s = R\theta$$

$$\frac{ds}{dt} = R\frac{d\theta}{dt}$$

$$\frac{d^2s}{dt^2} = R\frac{d^2\theta}{dt^2}$$

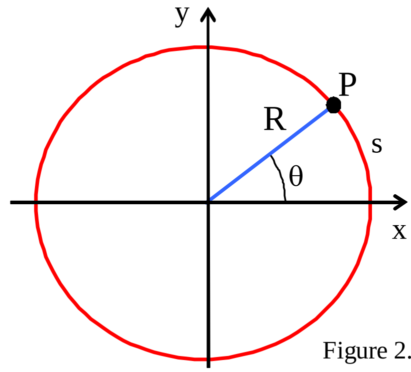


Figure 2.3

We define the quantity "angular velocity" ω by the relation $\omega = \frac{d\theta}{dt}$. In the uniform circular motion it is ω=constant by definition. We set $\omega=\omega_0$; as in (2.12) we get

$$\frac{d\theta}{dt} = \omega_0 \Rightarrow \theta = \omega_0 t + \theta_0 \tag{2.22}$$

In cartesian coordinates we have:

$$\begin{cases} x(t) = R\ \cos(\omega_0 t + \theta_0) \\ y(t) = R\ \sin(\omega_0 t + \theta_0) \end{cases} \tag{2.23}$$

$$\begin{cases} \dfrac{dx}{dt} = -\ \omega_0 R \sin(\omega_0 t + \theta_0) \\ \dfrac{dy}{dt} = \omega_0 R \cos(\omega_0 t + \theta_0) \end{cases} \tag{2.24}$$

$$\begin{cases} \dfrac{d^2x}{dt^2} = -\omega_0^2 R \cos(\omega_0 t + \theta_0) = -\omega_0^2\ x \\ \dfrac{d^2y}{dt^2} = -\omega_0^2 R \sin(\omega_0 t + \theta_0) = -\omega_0^2\ y \end{cases} \tag{2.25}$$

The functions sin and cos are periodic; with a period of 2π, so that they have the same value every time that t is increased by an amount $T = 2\pi/\omega_0$; T is called the "period"; another quantity related to this motion is the frequency, defined by the relation $\nu = 1/T = \omega_0/2\pi$

Harmonic motion

The point P moves of uniform circular motion; its projection on a diameter (for simplicity we take the one along the x axis) obeys to (2.23). By comparing (2.23) and (2.25) we have

$$\frac{d^2x}{dt^2} = -\omega_0^2 R \cos(\omega_0 t + \theta_0) = -\omega_0^2 x \tag{2.26}$$

The equation of motion is then

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \tag{2.27}$$

(2.27) is a linear homogeneous differential equation, with constant coefficients. Its solution is:

$$x(t) = A\sin(\omega t + \varphi) \tag{2.28}$$

with the time derivative:

$$\frac{dx(t)}{dt} = -\omega A\sin(\omega t + \varphi) \tag{2.29}$$

where A and φ are two integration constants that must be derived on the basis of the initial conditions. If we set $t = 0 \Rightarrow x(0) = x_0, \frac{dx}{dt} = 0$, from (2.29) we obtain $\varphi = 0$, while from (2.28) we get $A = x_0$

Chapter 3: Dynamics

NEWTON'S LAW OF MOTION

Newton's First Law of Motion: Every body continues in its state of rest or of uniform speed in a straight line unless it is compelled to change that state by a net force acting on it. This law is also called the *principle of inertia*.

Newton's Second Law of Motion: The acceleration of an object is directly proportional to the net force acting on it and is inversely proportional to its mass. The direction of the acceleration is in the direction of the applied net force.

Newton's Third Law of Motion: Whenever one object exerts a force on a second object, the second object exerts an equal and opposite force on the first.

THE FORCES

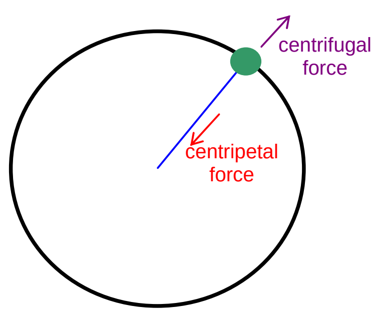
The Centrifugal Force

If you attach a string to a ball and swing it in a circular manner, then the force that is required to keep the ball moving in the circular path is called the **centripetal force**

$$ma_c = m\frac{v^2}{R} = m\omega^2 R \qquad (3.1)$$

and is directed inward, towards the axis of rotation

The **centrifugal force** is equal and opposite to the centripetal force: as you swing the ball with the string, you feel the string tug on you hand. The centrifugal force is not a real force, in fact it belongs to the category of the apparent forces, and is only an effect of the principle of inertia.



Gravitational force and Weight

Newton's Law of Universal Gravitation can be expressed as:

$$F_G = G\frac{m_1 m_2}{r^2} \qquad (3.2)$$

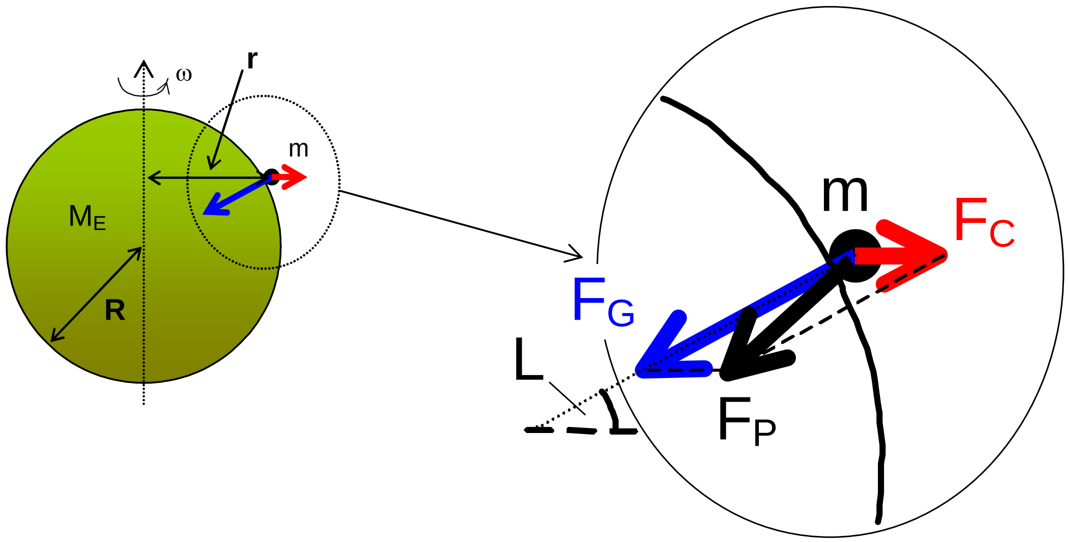
where F_G is the attractive force between m_1 and m_2 (in N), m_1 and m_2 are the masses in kg, and r is the distance between the masses, in metres.

The value of G, the gravitational constant is $6.67\cdot10^{-11}\ Nm^2kg^{-2}$. The numerical value of G depends on the fundamental units used.

The weight W is the sum of the gravitational force F_G (3.2)and the centrifugal force F_C (3.1), as shown in the figure. It is usually expressed as

$$\vec{W} = m\vec{g} \tag{3.3}$$

where m is the mass and $\vec{g}$ is the acceleration of gravity. The value of $\vec{g}$ is not constant, but in practice is assumed to be $9.8\,ms^{-2}$.



Putting together (3.2) and (3.1) (see figure) we have:

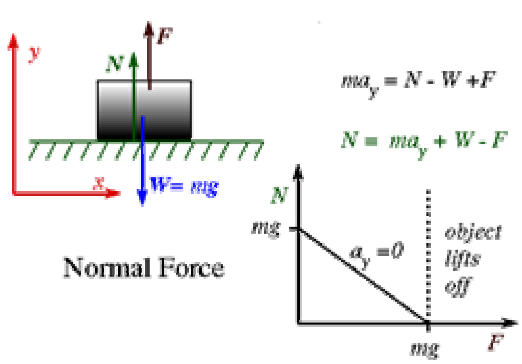
$$\left.\begin{array}{c} \vec{F}_G = \dfrac{GM_T m}{R_T^2}\hat{R}_T \\ \vec{F}_C = m\ \omega^2\vec{r} \end{array}\right\} \Rightarrow \vec{W} = \vec{F}_G + \vec{F}_C = \frac{GM_T m}{R_T^2}\hat{R}_T + m\omega^2\vec{r} \tag{3.4}$$

from which, comparing with (3.3), we find that

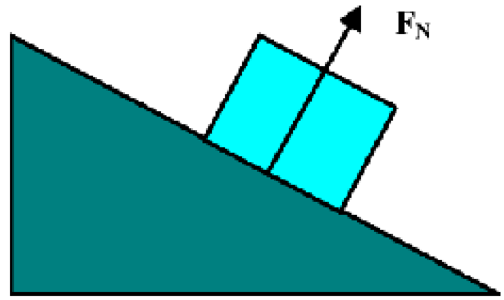
$$g = \frac{|W|}{m} = \left[\left(\frac{GM_T}{R_T^2}\right)^2 + \left(\omega^2 r\right)^2 - 2\left(\frac{GM_T}{R_T^2}\right)\left(\omega^2 r\right)\right]^{\frac{1}{2}} \tag{3.5}$$

Normal force

The normal force on a body is generally associated with the force that the surface of one body exerts on the surface of another body in the absence of any frictional forces between the two surfaces. The normal force is an action-reaction force. A surface will not exert a normal force on an object in contact with it unless some other external force pushes the object into the surface.



The atoms in the surface are compressed microscopically to create the normal force. The surface deforms imperceptibly and produces a reaction force equal to the force pressing the object into the surface. For an object sitting on a horizontal surface, the normal force will be equal to the weight of the object.

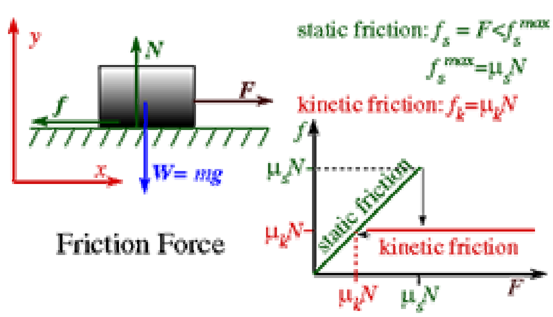


It is important to remember that the normal force is always perpendicular to the surface. This is the origin of its name - normal to the surface (*from the Latin word norma which meant carpenter's square and indicating perpendicularity)*. If the surface is not horizontal than the normal force will not point upwards but in whatever direction is perpendicular to the surface. The sketch shows the normal force exerted on a block by an incline.

Friction

Friction is the resistive force of a surface against an object. There are two different kinds of friction: static and kinetic. Kinetic friction is described by the equation

$$f_k = \mu_k N \tag{3.6}$$



The coefficient of friction μ_k is a constant that is determined from experiments. It is a measure of how rough the floor is. A very rough floor will have a high coefficient of friction. N is the "normal force", or the force of the floor on the box. Thus, friction depends on two things: the normal force (N) and the roughness of the floor (μ_k).

If the box was not in relative motion with respect to the roof, i.e. in the "static" case, we may image the molecules pulling or pushing on each other at the borders of the contact surface. Think of a sort of weak chemical bond between the box molecules and the roof molecules. The bonds are elastic and, like a spring, the size of the force depends on the amount of stretch or compression. Experiment shows that for each pair of surfaces there is some maximum force that these weak molecular bonds can supply. If the box is sitting on the roof, we can lift it off, breaking the bonds completely or we can push or pull it so that it slides, essentially breaking only the "parallel-to-the-roof" component of the bonds and reverting to the "collision" mode described in the kinetic friction discussion. On way to observe the transition from static friction to kinetic friction is to place the box on a board and then raise one end of the board. As the slope of the board increases, the box stays put until some maximum angle in reached. Careful observation of the box's subsequent motion shows that if the board remains at this angle, the box then accelerates down the inclined board. Newton's first law of motion tells us that as the board was being slowly raised, the sum of the forces on the box remained zero until the maximum angle was reached. Since the component of the gravitational force parallel to the board increases with q, (it is mg sinθ), there must be an opposing component of the molecular bond force of the same size to keep the sum of the forces parallel to the board equal to zero. We call this component the force of static friction, f_s. Once the slope angle reaches θ_{max}, f_s has reached its maximum possible value f_s^{max}. That is, the molecular bonds parallel to the board have stretched as far as they can. Any slight increase in angle will break these bonds and the box will slide.

Drag

The force that resists the motion of an object through a fluid. Drag is directed opposite the direction of motion of the object relative to the fluid. For velocities $v \le 2\, m\, s^{-1}$ the expression of the force is

$$F_d = -b\vec{v} \tag{3.7}$$

where b is a factor which depends on the object section perpendicular to $\vec{v}$, the shape of the object, and on the fluid’s density and viscosity.

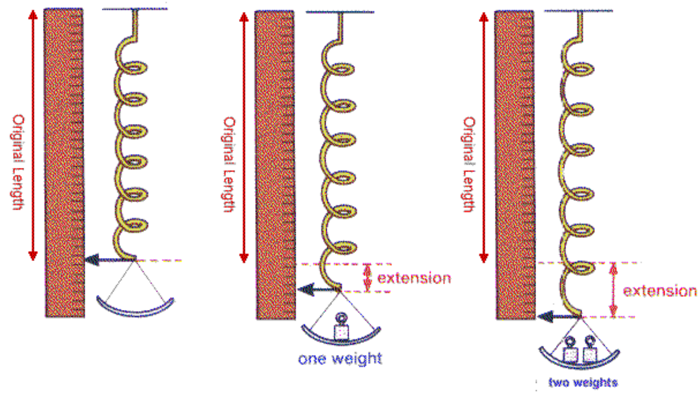
Consider hydrodynamic flow characterized by a fluid of some mass density, flowing at some velocity, past an object of some length. The Reynolds number of the flow past the object (or `wake') is the dimensionless combination

$$\mathrm{Re} = \frac{\text{velocity}\times\text{length}\times\text{mass density}}{\text{viscosity}} \tag{3.8}$$

Alternately, the length used to estimate Re might be the width of a channel (or pipe) that fluid is flowing through.

If Re >> 1 then the flow is *turbulent*, with complicated vortices and eddies at a wide range of sizes which continually are reorganizing. Turbulence is not well understood at all, and will not be discussed further. Fortunately turbulence is largely irrelevant at molecular and cellular scales. The velocities and lengths we will worry about end up just being too small to ever allow Re to get big. For sufficiently tiny things moving sufficiently slowly, Re << 1.

Elastic force



Elastic forced is described by the **Hooke's Law**. When we apply a force to a spring, it stretches. If we apply double the force, it stretches twice as much, so long as we don't over-do it. We measure the original length of the spring when we start. When it stretches, we measure the extension; that's how much longer it is than it was when we started. Hooke's Law is written:

$$F_e = -k\Delta x \qquad (3.9)$$

where k is the elastic constant, measured in (N m^{-1}) and Δx the displacement from the equilibrium position. The Hooke's law states:

- the extension is proportional to the force
- the spring will go back to its original length when the force is removed

Momentum

Momentum is a fundamental quantity in mechanics that is conserved in the absence of external forces. For a single particle of mass *m* with velocity **v**, the momentum is defined as

$$\vec{P} = m\vec{v} = m\frac{d\vec{r}}{dt} \qquad (3.10)$$

Newton's second law was initially written

$$\vec{F} = m\vec{a} \qquad (3.11)$$

In the case that the mass is constant, then

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = \frac{d\left(m\vec{v}\right)}{dt} = \frac{d\vec{P}}{dt} \qquad (3.12)$$

So Newton's second law can be cast in the form: The rate of change of momentum of an object is directly proportional to the net force acting on it. The direction of the time derivative of momentum is in the direction of the applied net force.

Conservation of momentum

From Newton's second law, a force $\vec{F}$ produces a change in momentum

$$\vec{F} = \frac{d\vec{P}}{dt} \qquad (3.13)$$

Therefore, if $\vec{F} = 0$, then

$$\frac{d\vec{P}}{dt} = 0 \tag{3.14}$$

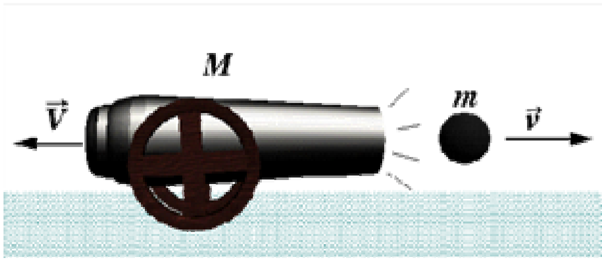
And

$$\vec{P} = const \tag{3.15}$$

so the particle has constant velocity.

Example of conservation of momentum: cannon and bullet

We consider a cannon, free to move, firing a ball. We can think the system constituted from two elements: the cannon of mass M and the ball of mass m. At $t = 0$ the cannon is at rest, so that the system possesses an initial velocity equal to zero, and therefore initial momentum equal to zero. The forces that generate the outcome of ball are all inner to the system cannon + ball, and therefore the sum of the external forces is null.



Momentum is conserved; therefore the quantity of motion of the system, at the instant of firing, must be zero as well. The ball is expelled with velocity $\vec{v}$. The equation of conservation of momentum is

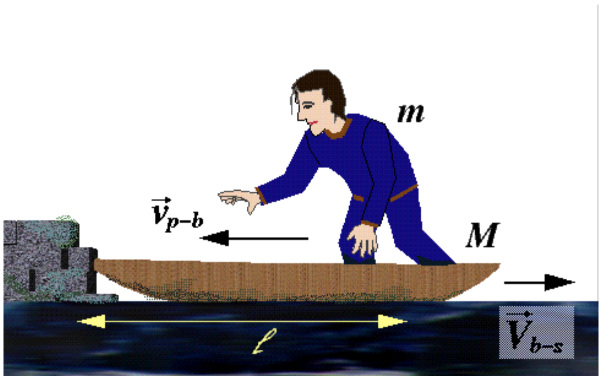
$$0 = m\vec{v} + M\vec{V} \tag{3.16}$$

where $\vec{V}$ is the velocity of the cannon. From (3.16) one sees that the cannon acquires a velocity directed in the direction opposite to that one of the ball, and such that the relationship between the two velocities is equal to the inverse of the relationship of the masses:

$$V/v = m/M$$

Examples of conservation of momentum: man on the boat

A person of mass m finds on a boat of mass M and length L, near the end more distant from the shore, and he begins to walk on the boat. The velocity of the person relatively to the shore $(\vec{v}_{ps})$ is equal to the combination of the velocity of the boat with respect to the beach $(\vec{V}_{bs})$ and the velocity of the person with respect to the boat $(\vec{v}_{pb})$:



$$\vec{v}_{ps} = \vec{v}_{pb} + \vec{V}_{bs} \tag{3.17}$$

Initially the system boat + person has zero momentum because both the person and the boat have zero velocity: $\vec{P}_i = 0$. From the conservation of momentum:

$$\vec{P}_f = m\vec{v}_{ps} + M\vec{V}_{bs} = 0 \tag{3.18}$$

$$m\left(\vec{v}_{pb} + \vec{V}_{bs}\right) + M\vec{V}_{bs} = 0$$

$$\vec{V}_{bs} = -\frac{m}{m+M}\vec{v}_{pb} \tag{3.19}$$

The boat is moved from the beach! How much goes away? The person walks for a given time Δt given by

$$\Delta t = \frac{L}{v_{pb}} \tag{3.20}$$

In the same interval of time the boat goes away from the shore a distance

$$\Delta x = v_{bs}\Delta t = L\frac{v_{bs}}{v_{pb}} = -\frac{m}{m+M}L \tag{3.21}$$

that is calculated by employing (3.20) and (3.19).

EXAMPLES INVOLVING FRICTION AND DRAG

Friction is often treated as an annoying phenomenon, which causes big troubles to students who must solve problems of mechanics. Anyway, it is very important and must be studied carefully. In fact, no real system is completely frictionless, although we will usually treat systems as if they were. For instance, if there were no friction between your shoe and the floor, you could not walk! The friction in this case is a force in the opposite direction of your movement which keeps you from slipping.

Friction often presents itself as drag, when an object is moving through a fluid medium. It is a force proportional to the object's velocity, but opposite to the direction of motion:

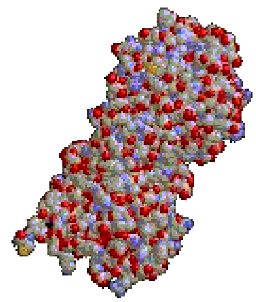
$$\vec{F}_d = -b\vec{v} \tag{3.22}$$

Here b is the "drag coefficient", with dimensions of mass over time. The drag coefficient depends on the radius and shape of the object and the viscosity ("stiffness") of the medium. For a sphere of radius r, the drag coefficient is

$$b_0 = 6\pi\eta R \tag{3.23}$$

where η is the viscosity of medium in g / cm s. For other shapes, we multiply by a "shape factor " b/b_0 which depends on the shape of the object. So our usual procedure will be to start with the volume of an object, assume it is a sphere and compute its radius and b_0, and then multiply by its shape factor to obtain its drag coefficient. For instance, for a rod-like molecule such as Immunoglobulin G, the shape factor is approximately 1.5. If its volume is $2\cdot 10^{-19}$ cc and it is falling through a fluid of viscosity 0.02 g / cm s, the radius of an equivalent sphere would be

$$r = \left(\frac{3V}{4\pi}\right)^{1/3} = 3.63\cdot 10^{-7}\,cm$$



The drag coefficient of an equivalent sphere would be

$$b_0 = 1.37\cdot 10^{-7}\,gs^{-1}$$

and the drag coefficient of the molecule would be

$$b = 1.5\; b_0 = 2.05\cdot 10^{-7}\; g/s$$

We can also write the drag coefficient of a substance in terms of the temperature and the "diffusion constant" of the substance:

$$b = \frac{k_B T}{D}$$

Here $k_B = 1.38\cdot 10^{-23}\; J/K$ is the Boltzmann constant, T is the temperature in Kelvin (which is the temperature in Celsius + 273.15) and D is the diffusion constant in cm^2/s. The diffusion constant measures the rate at which something diffuses through a medium; it depends on the substance, the medium and the temperature. In this form, the shape and viscosity information is "hidden" in the diffusion constant.

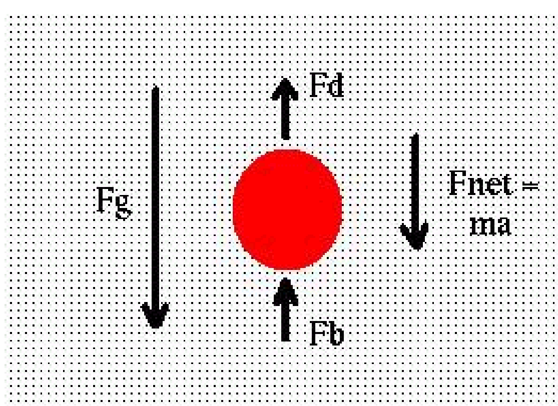
Consider an object slowly sinking in a swimming pool. It sinks due to the attractive force of gravity:

$$\vec{F}_g = m\vec{g}$$

(here the velocity is considered negative, and $g = -9.8\; ms^{-2}$). Opposing gravity, however, is the buoyant force of the water, which attempts to raise the object. The buoyant force is equal to the weight of the fluid displaced by the object:

$$\vec{F}_b = -mV_S\rho g$$

Here V_S is the specific volume of the object (volume per unit mass), ρ is the density of the medium, and g is the gravitational acceleration. Thus mV_S gives



the volume of the object, and ρg gives the gravitational force per unit volume of the fluid. Since g is negative, the minus sign indicates that the buoyant force is directed upwards.

Equating the sum of the forces due to gravity, buoyancy and drag to the mass times the acceleration we find

$$\vec{F}_T = \vec{F}_g + \vec{F}_b + \vec{F}_d = m\vec{a} \tag{3.24}$$

The projection of (3.24) on the vertical axis is:

$$mg - mV_S \rho g - bv = ma \tag{3.25}$$

or

$$mg(1 - V_S \rho) - bv = ma \tag{3.26}$$

Given the object's mass, specific volume and drag coefficient (m, v and b), as well as the acceleration due to gravity and the density of the fluid (g and η), we could solve for the acceleration as a function of velocity, or vise versa.

Since the acceleration is the time rate of change of velocity, the equation can be written in terms of the velocity alone:

$$mg(1 - V_S \rho) - bv = m\frac{dv}{dt} \tag{3.27}$$

This means (the calculations are reported in the appendix) that the equation has a solution whose velocity is given by

$$v(t) = (1 - V_S \rho)\left(\frac{mg}{b}\right)\left(1 - e^{-\frac{b}{m}t}\right) \tag{3.28}$$

For "large" times relative to m/b, that is, when $t \gg m/b$ the exponential approaches zero and the velocity approaches the "terminal velocity":

$$v_\infty = (1 - V_S \rho)\left(\frac{mg}{b}\right) \tag{3.29}$$

It is also possible to define a "sedimentation constant"

$$s = \frac{v_\infty}{g} = (1 - V_S \rho)\left(\frac{m}{b}\right) \tag{3.30}$$

which depends only on the characteristics of the object and the fluid, but not on the acceleration due to gravity. This is helpful because gravity is not the only external force which we might want to consider. The above results work for the object sinking in the swimming pool, or an object falling through the air, where g in both cases is due to gravity, but also for an object in a centrifuge.

THE CENTRIFUGE

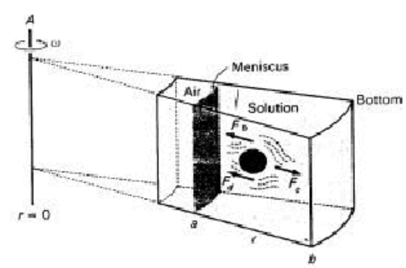
The centrifuge is an essential instrument in cell and molecular biology research. It is primarily used to separated biological components based upon differential sedimentation properties. Many types of centrifuges are available for various applications. All centrifuges basically consist of a motor which spins a rotor containing the experimental sample. The differences between centrifuges are in the speeds at which the samples are centrifuged and the volumes of samples.

Sedimentation theory

Spinning around an axis creates a centrifugal field (Figure). A particle in a centrifugal field will experience a centrifugal force defined by:

$$F_c = m\omega^2 r \tag{3.31}$$

where F_c is the centrifugal force, m the mass of the particle, ω the angular velocity and r the distance from the axis. This force will be opposed by a buoyant force (F_b) and a frictional force (F_d).The buoyant force represents the force it takes to displace solvent as the particle moves through the centrifugal field. The frictional force represents the drag on the particle as it passes solvent molecules. These two forces are respectively defined as:



$$\vec{F}_b = -m\omega^2\vec{r} \text{ and } \vec{F}_b = -b\vec{v}$$

where m_0 is the mass of the displaced solution, b the frictional coefficient and v the velocity of the particle. The particle will move at a velocity such that the total force equals 0, thus:

$$\vec{F}_c + \vec{F}_b + \vec{F}_d = 0$$

or

$$m\omega^2 r - m_0\omega^2 r - bv = 0 \tag{3.32}$$

substituting $mV_S\rho_S = m_0$, where V_S = partial specific volume of the particle and ρ_S = density of the solvent, and solving for v results in:

$$v_\infty = \frac{\omega^2 rm\left(1 - V_S\rho_s\right)}{b} = \frac{\omega^2 rmV_S\left(\rho_p - \rho_s\right)}{b} \tag{3.33}$$

This equation (expressed in the two different forms) tells us several things about sedimentation:

1. The more massive a particle, the faster it moves in a centrifugal field
2. The denser a particle (i.e., the smaller its V_S) the faster it moves in a centrifugal field.
3. The denser the solution, the slower the particle will move in a centrifugal field.

4. The greater the frictional coefficient (factors such as viscosity, particle shape, etc. influence this parameter), the slower the particle will move.
5. The particle velocity is 0 when the solution density is greater than the particle density.
6. The greater the centrifugal force $\left(\omega^2 r\right)$ the faster the particle sediments.

The velocity per unit force will be defined as the sedimentation coefficient (s), or:

$$s = \frac{v}{\omega^2 r} = \frac{m\left(1 - V_S \rho_s\right)}{b} \tag{3.34}$$

when mass is expressed in g and *b* in g/sec s ranges from 10^{-13} to 10^{-11} sec. This is normally expressed in Svedberg (S) units where 1 S = 10^{-13} s. The higher the S value, the faster that particle will sediment.

In the study of cell organelles, different fractions of the subcellular particles are routinely separated with a centrifuge. It was by this type of isolation technique that mitochondria were discovered to be responsible for the entire tricarboxylic acid cycle and ribosomes accountable for protein biosynthesis.

The sedimentation coefficient is determined by measure the velocity of a particle in a known centrifugal field. S units need to be normalized according to the composition of the medium. Sedimentation coefficients are usually determined with an analytical ultracentrifugation, also referred to as a model E ultracentrifuge. The rotor in the model E centrifuge allows the sample to be monitored spectrophotometrically as the sample is centrifuging. As the sample migrates in the centrifugal field the absorbance across the chamber will change. The sedimentation rate can be calculated from this change in absorbance. In addition, physical characteristics (such as size, density and shape) of a particle, or molecule, can be determined from the sedimentation rate of a particle in a medium of known composition.

The Relative Centrifugal Force (RCF) is defined as the ratio of the centrifugal force to the force of gravity, or:

$$RCF = \frac{F_c}{F_g} = \frac{\omega^2 r}{980}$$

where ω is expressed in radians/sec. ω can be converted to revolutions per minute (rpm) by

substituting $\omega = \frac{2\pi(rpm)}{60}$

resulting in:

$$RCF = 1.119 \cdot 10^{-5} (rpm)^2 r$$

where r is expressed in cm. RCF units are expressed as "*x* g".

In a typical separation scheme, an isotonic 0.25-0.35M sucrose solution is mixed with cells. The mixture is placed in a bead homogenizer, an ultrasonic cell disrupter, or a simple kitchen blender and the cell membranes are broken to spill out the cell contents. The resulting mixture of subcellular organelles can be placed in a centrifuge and spun at 1000*g* for 10 minutes. Intact cells and heavy nuclei are collected at the bottom of the centrifuge tube as a pellet. After the supernatant is further centrifuged at 10,000*g* for 20 minutes, subcellular particles of intermediate terminal velocities such as mitochondria, lysosomes, and microbodies may be collected. The smaller and lighter particles (ribosomes, endoplasmic reticulum fragments, cell membranes, and microsomes) can be further separated from the supernatant of the preceeding stage by centrifugation at 100,000*g* for 60 minutes. The final supernatant may be considered to be the soluble portion of the cell cytoplasm. Notice the increasingly longer time and especially the exponentially increasing centrifugation speed required to effect separation. The last speed is achievable in an ultracentrifuge. This mode of separation is commonly called *differential centrifugation.*

Each fraction obtained through differential centrifugation contains quite a few different types of organelles which have similar sedimentation velocities. Because this factor is a combination of both the size and the density, the fraction can be further separated based on density alone irrespective of the sizes. This second stage can be accomplished by a process known as *density gradient centrifugation.*

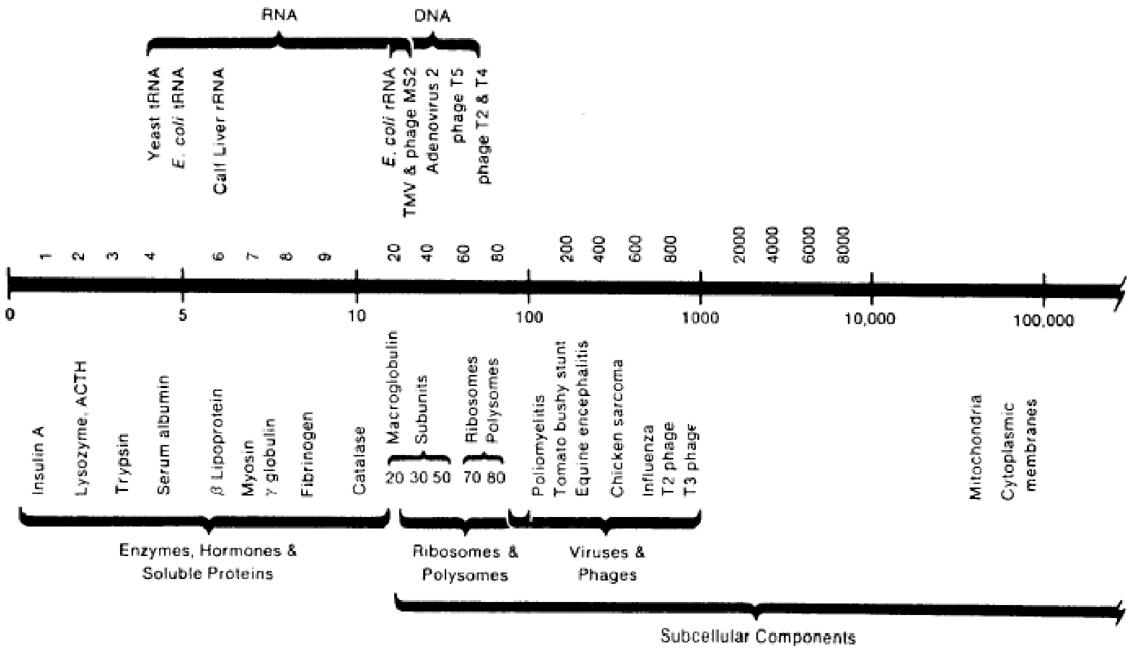
The density gradient may be established naturally by simply placing sucrose crystals at the bottom of a test tube. The sugar dissolves in the solution and diffuse toward the top. However, the time required to establish a sugar gradient in this manner is unacceptably long, and the process is not well regulated. A more practical method of establishing a density gradient is by placing layer after layer of sucrose solutions of different concentrations, thus, densities, in a test tube, with the heaviest layer at the bottom and the lightest layer at the top. The cell fraction to be separated is placed on top of the layer. A particle will sink if the density of the particle is higher than that of the immediate surrounding solution. It will continue to sink until a position is reached where the density of the surrounding solution is exactly the same as the density of the particle. A centrifuge can be highly helpful to accelerate this process of reaching the quasi-equilibrium point; however, unlike the differential centrifugation technique used during the first stage of cell separation, the length of centrifugation for this second stage does not matter too much, as long as the system is permitted to come to quasi-equilibrium.

Organelle	Diameter (μm)	Density (g/cm^3)
Nuclei	5-10	1.4
Mitochondria	1-2	1.1
Ribosomes	0.02	1.6

Lysosomes	1-2	1.1

Size and density of some typical organelles

For example, subcellular compartments and macromolecules have sedimentation coefficients which reflect their size, shape and density (see the figure).



Work and energy

WORK AND ENERGY

Work

Work is a function defined as:

$$W = \vec{F} \cdot \Delta\vec{r} \quad (4.1)$$

where $\Delta\vec{r}$ is the displacement. We define also an elementary work

$$dW = \vec{F} \cdot d\vec{r} \quad (4.2)$$

in correspondence to an elementary displacement. Work is measured in the International System in Joule:

$$1J = 1N \cdot 1m$$

Kinetic energy

The function Kinetic energy is defined as

$$K = \frac{1}{2} m v^2 \quad (4.3)$$

Theorem of kinetic energy

The theorem of kinetic energy states that *the variation in the Kinetic energy between two states 1 and 2 equals the total work done by all the forces acting on the system between the same two states.*

The II Newton's law:

$$\vec{F} = \frac{d\vec{P}}{dt}$$

can be written, for constant mass:

$$\vec{F} = m\frac{d\vec{v}}{dt} \quad (4.4)$$

For simplicity we shall demonstrate the theorem in one dimension. The projection of (4.4) is

$$F = m\frac{dv}{dt} \quad (4.5)$$

and the elementary work is

$$dW = F\,dx = m\frac{dv}{dt}dx = m\frac{dx}{dt}dv = mv\,dv \quad (4.6)$$

Now we can integrate:

$$\int_1^2 dW = \int_1^2 mv\,dv \tag{4.7}$$

Here we compute the definite integrals between two states 1 and 2, for which we have: in 1 $v = v_1$ and in 2 $v = v_2$. We obtain:

$$W_{12} == m\int_1^2 v\,dv = \frac{1}{2}m\left[v_2^2 - v_1^2\right] = K_2 - K_1 \tag{4.8}$$

Potential energy and Conservative forces

If it happens that the work a force does on an object in moving it from A to B is path independent, i.e. it depends only on the end points of the motion, then it is possible to write this work as a difference between the values a function takes at the end points of the motion:

$$W_c = F_c(x_B - x_A) = U_A - U_B \tag{4.9}$$

The function U is called **Potential Energy**, and the force F is called **conservative**. Examples: the force of gravity and the spring force are conservative forces. The work done by an external conservative force is stored as some form of potential energy.

Conservation of mechanical energy

From the theorem of kinetic energy

$$W = \Delta K = K_2 - K_1 \tag{4.10}$$

By the definition of potential energy

$$W = -\Delta U = U_1 - U_2 \tag{4.11}$$

equating (4.10) and (4.11)

$$\Delta K = -\Delta U \tag{4.12}$$

that can be rewritten

$$\Delta K + \Delta U = \Delta\left(K + U\right) = \Delta E = 0 \tag{4.13}$$

$\Delta E = 0$ means that

$$E = K + U \tag{4.14}$$

and, by the fact that the states 1 and 2 have been arbitrarily chosen, follows

$$E = \text{constant} \tag{4.15}$$

Work of non-conservative forces

For a **non-conservative** (or dissipative) force, the work done in going from A to B depends on the path taken. Examples: friction and air resistance. We have

$$W_T = K_2 - K_1 = \Delta K \tag{4.16}$$

The total work can be split between the work done by the sum of the conservative forces, W_C and the work done by the sum of the non conservative forces, W_{NC}. The first is

$$W_C = U_1 - U_2 = -\Delta U \tag{4.17}$$

so that (4.16) becomes

$$W_T = W_C + W_{NC} = -\Delta U + W_{NC} = \Delta K \tag{4.18}$$

from which we see that

$$W_{NC} = \Delta U + \Delta K = \Delta\left(U + K\right) = \Delta E \tag{4.19}$$

or, in other words,

$$W_{NC} = E_2 - E_1 \tag{4.20}$$

which means that the total variation of the mechanical energy is equal to the work done by the non conservative forces.

THE HARMONIC OSCILLATOR

The equation of motion

In the figure 1 a mechanical system formed of a spring, whose elastic constant is k, and a mass m. The forces acting on the system are the weight $m\vec{g}$, the normal force $\vec{N}$ and the elastic force which can be written $-kx\hat{i}$, where x is the displacement of the mass from its equilibrium position. The equation expressing the 2nd principle of mechanics $\left(\vec{F} = m\vec{a}\right)$ is then:

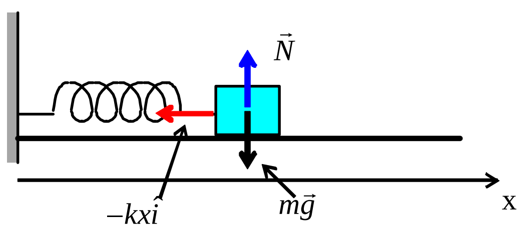


Figure 1: the harmonic oscillator

$$-kx\hat{i} + m\vec{g} + \vec{N} = m\vec{a} \tag{4.21}$$

The projection of (4.21) on the x axis is:

$$-kx = m\frac{d^2x}{dt^2} \tag{4.22}$$

This equation admits solution of the type:

$$x(t) = A\cos\left(\omega t + \varphi\right) \tag{4.23}$$

The first and second order time derivatives of (4.23) are, respectively:

$$\frac{dx(t)}{dt} = -A\omega\sin\left(\omega t + \varphi\right) \tag{4.24}$$

$$\frac{d^2x(t)}{dt^2} = -A\omega^2 \cos\left(\omega t + \varphi\right) \tag{4.25}$$

We now set the initial conditions: for $t = 0$ it is $x\left(0\right) = x_0$ and $\left.\frac{dx}{dt}\right|_{x=0} = 0$. From (4.24) it follows:

$$\left.\frac{dx}{dt}\right|_{x=0} = -A\omega \sin\left(\varphi\right) = 0 \Leftrightarrow \varphi = 0 \tag{4.26}$$

whilst from (4.23), using also(4.26), it follows

$$x(0) = A\cos(0) = x_0 \Leftrightarrow A = x_0$$

Substituting the values obtained for A and φ in (4.23), (4.24) and (4.25) we obtain:

$$x(t) = x_0 \cos\left(\omega t\right) \tag{4.27}$$

$$\frac{dx(t)}{dt} = -\omega x_0 \cos\left(\omega t\right) \tag{4.28}$$

$$\frac{d^2x(t)}{dt^2} = -\omega^2 x_0 \cos\left(\omega t\right) \tag{4.29}$$

The substitution of (4.27) and (4.29) in (4.22) gives:

$$-k + \omega^2 m = 0 \tag{4.30}$$

This is an second grade algebraic equation whose solution are $\omega = \pm\sqrt{k/m}$, i.e. the system has a characteristic angular frequency $\omega_0 = \sqrt{\frac{k}{m}}$, corresponding to a frequency $\nu_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ and a period $T = \frac{1}{\nu_0} = 2\pi\sqrt{\frac{m}{k}}$.

Energy of the harmonic oscillator

The potential energy of the system is $U = \frac{1}{2}kx^2$; the substitution of (4.27) gives

$$U = \frac{1}{2}kx^2 = \frac{1}{2}k\left[x_0 \cos\left(\omega_0 t\right)\right]^2 \equiv \frac{1}{2}kx_0^2 \cos^2\left(\omega_0 t\right) \tag{4.31}$$

and the kinetic energy $K = \frac{1}{2}mv^2 = \frac{1}{2}m\left[\frac{dx}{dt}\right]^2$, by using (4.28), becomes

$$K = \frac{1}{2}m\left[\frac{dx}{dt}\right]^2 = \frac{1}{2}m\left[-x_0\omega_0 \sin\left(\omega_0 t\right)\right]^2 = \frac{1}{2}m\omega_0^2 x_0^2 \sin^2\left(\omega_0 t\right) \tag{4.32}$$

From (4.30) we get $k = m\omega_0^2$, so that (4.32) can be rewritten

$$T = \frac{1}{2}kx_0^2 \sin^2(\omega_0 t) \qquad (4.33)$$

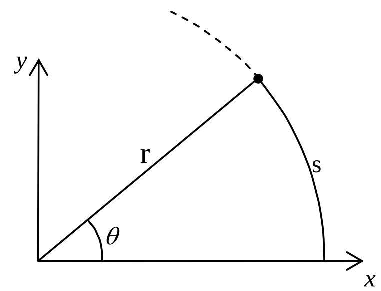
The sum of kinetic and potential energies is thus, by using (4.31) and (4.33):

$$E = K + U = \frac{1}{2}kx_0^2 \sin^2(\omega_0 t) + \frac{1}{2}kx_0^2 \cos^2(\omega_0 t) = \frac{1}{2}kx_0^2 \quad \forall \ t \qquad (4.34)$$

which shows that the total mechanical energy is constant.

ANGULAR QUANTITIES

A particle travelling around the circumference of a circle has a well defined (though constantly changing!) position



$$s = r\theta$$

which is just the arc length traversed by the particle. Its velocity and acceleration are defined as before. Since the radius is a constant, this means that

$$v = \frac{ds}{dt} = r\frac{d\theta}{dt}$$

which defines the "angular velocity" (rate of change of angle)

$$\omega = \frac{d\theta}{dt}$$

and

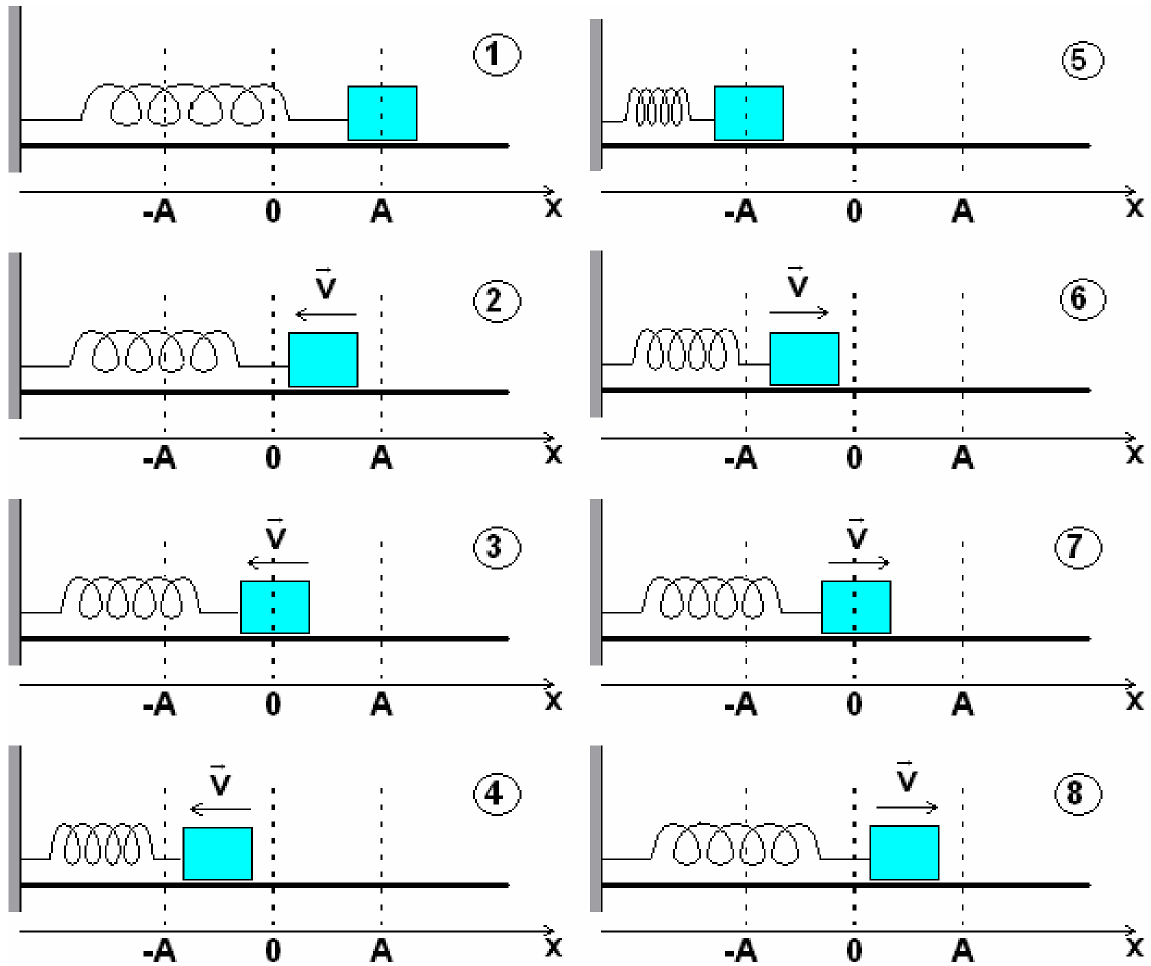
$$a = \frac{dv}{dt} = r\frac{d^2\theta}{dt^2}$$

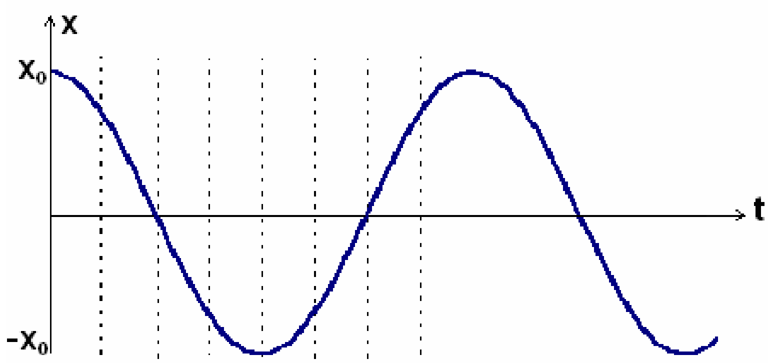
which defines the "angular acceleration" (rate of change of angular velocity)

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

Substituting these equations into the equations for uniform accelerated motion, and dividing through by the common factor r, we find angular analogs which are isomorphic to the linear uniform accelerated motion equations:

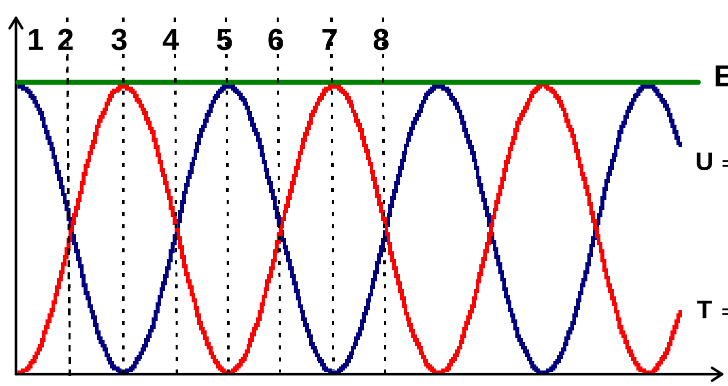
$$\theta = \theta_0 + \omega t + \frac{1}{2}\alpha t^2 \text{ and } \omega = \omega_0 + \alpha t$$





THE HARMONIC OSCILLATOR

(Top) The figure describes the various states of an harmonic oscillator.
(Middle) The position of the oscillator vs. time.
(Bottom) The potential (blue) and kinetic (red) energy of the harmonic oscillator vs. time. The numbers refer to the states of the oscillator illustrated in the top part of the figure.



$$E = T + U$$

$$U = \frac{1}{2} k x_0^2 \cos^2 \omega t = U_{max} \left[\frac{1 + \cos 2\omega t}{2} \right]$$

$$T = \frac{1}{2} k x_0^2 \sin^2 \omega t = T_{max} \left[\frac{1 - \cos 2\omega t}{2} \right]$$

This is possible because motion on a circle is essentially a one dimensional problem, and angle is just as good a variable as distance. Note that the definition for centrifugal acceleration above is just α here (ν was rate of rotation in revolutions per unit time, while ω is radians per unit time).

In addition, if we define the "moment of inertia" of a point particle (which in rotational motion acts as mass) as

$$I = mr^2$$

we can define the rotational kinetic energy as

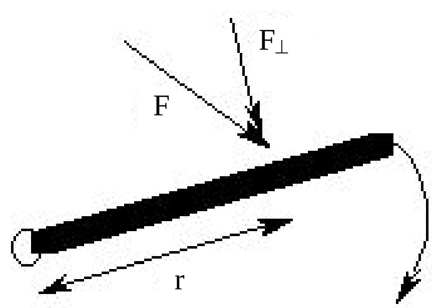
$$K = \frac{1}{2} I\omega^2$$

which is isomorphic to its linear analog. Note that an object in general can have "translational" (linear) as well as rotational kinetic energy; its kinetic energy is the sum of the two. The angular dynamical variables are "angular momentum" and "torque":

$$L = I\omega \text{ and } \tau = I\alpha = mar = F_{\perp} r$$

Note that the units of angular momentum and torque are different from their linear counterparts. In particular, the dimensions of torque are force times length. Just as force is necessary to change linear velocity, torque is necessary to change angular velocity. The length is called the "lever arm", and represents the distance from the center of rotation to the point of application of the force which causes the torque.

Compare the effort required to hold your bookbag in your hand, with that required to hang it from the middle of your arm. It is harder to hold in your hand because the lever arm is longer, and thus the same weight applies more torque, for which your bicep must compensate.



In the penultimate equation above, which gives the torque in the form which we will most often use, $F_{\perp}$ is the component of the force causing the rotation which is perpendicular ("normal") to the rotating object, and r is the lever arm.

If we resolve the force into components parallel and normal to the the rotating body, we see at once the the force parallel to the body cannot affect its rotation. Thereforce, there are two conditions under which a force cannot cause rotation:

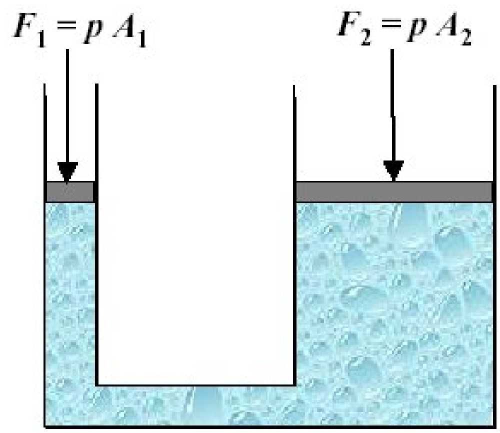
1) it is applied at the pivot (the lever arm is zero), or
2) it is parallel to the object.

Fluids

STATICS

Pressure

Pressure is defined as force per unit area. It is usually more convenient to use pressure rather than force to describe the influences upon fluid behavior. The standard unit for pressure is the Pascal, which is a Newton per square meter. The basic law about pressure is **Pascal's law**: it states that when there is an increase in pressure at any point in a confined fluid, there is an equal increase at every other point in the container. An application of this law is the so-called hydraulic press: a multiplication of force can be achieved by the application of fluid pressure according to Pascal's principle, which for the two pistons implies



$$P_1 = P_2 \tag{5.1}$$

From (5.1), equating the pressure in the two parts of the press in figure, we have

$$F_2 = F_1 \frac{A_2}{A_1} \tag{5.2}$$

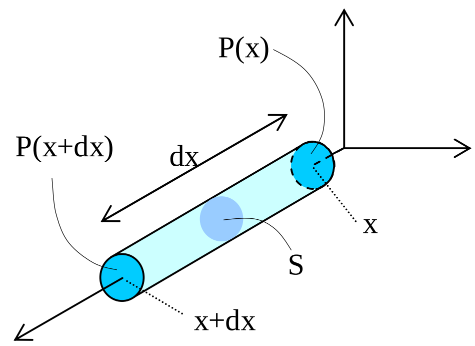
This allows the lifting of a heavy load with a small force, as in an auto hydraulic lift, but of course there can be no multiplication of work.

Hydrostatic pressure, buoyancy, Archimede's principle

Any body which, when placed in a fluid and which displaces a volume of the same fluid, will experience an upwards buoyancy force. This buoyancy force will be equal to the mass of the displaced fluid (density multiplied by volume) multiplied by the acceleration due to gravity. For instance, a boat will displace a volume of water which will supply an upwards buoyancy force to balance the weight of the boat. This is called the **Archimede's principle.**

The equations of static of fluids

The equations that describe fluids at rest follow from the **linear momentum principle**, applied to a fixed mass of fluid: a **material volume.** In general, forces can be of two types, **body forces** and **surface forces**. So we require (remember, it's at rest) that the **rate of change of momentum** be equal to zero. We study the equation along the x axis. Take a cylinder, of base S and length dx, with its axis oriented along the x axis of a reference frame. The two bases of the cylinder are at x and (x+dx). On the two bases act the following surface forces: on the base at x a force whose x component is



$$F_1 = P(x)S \tag{5.3}$$

while on the base at x+dx a force whose x component is

$$F_2 = -P(x+dx)S \tag{5.4}$$

where the minus sign comes from being the force $\vec{F}_2$ directed in the negative x direction. The pressure at point x+dx is related to the pressure at point x by the relation:

$$P(x+dx) = P(x) + \left(\frac{dP}{dx}\right)dx \tag{5.5}$$

On the mantle the pressure forces have no components in the x direction, and can thus be neglected. The body force (gravity, centrifugal force) per unit mass $\vec{G}$ has components (X, Y, Z), so that the body force along the x axis is

$$F_{Vx} = \rho XSdx \tag{5.6}$$

where Sdx is the cylinder volume. The condition that the total force must be equal to zero con be expressed as

$$F_1 + F_2 + F_{vx} = P(x)\,S - P(x+dx)S + \rho XSdx = 0 \tag{5.7}$$

Substituting (5.5) in (5.7) gives:

$$P(x)\,S - \left[P(x) + \left(\frac{dP}{dx}\right)dx\right]S + \rho XSdx = 0 \tag{5.8}$$

(5.8) can be simplified:

$$\cancel{P(x)S} - \cancel{P(x)S} + \left(\frac{dP}{dx}\right)\cancel{Sdx} + \rho X\ \cancel{Sdx} = 0 \tag{5.9}$$

so that we finally have:

$$\frac{dP}{dx} = \rho X \tag{5.10}$$

We find other two equations for the y and z components, and the three together are the equations of static for fluids:

$$\begin{cases} \dfrac{dP}{dx} = \rho X \\ \dfrac{dP}{dy} = \rho Y \\ \dfrac{dP}{dz} = \rho Z \end{cases} \tag{5.11}$$

Stevin's law

In the case of gravity the body force is

$$\vec{G} = -g\hat{k} \tag{5.12}$$

having chosen a z axis oriented upwards. Form the third equation of (5.12) we have:

$$dP = -g\rho dz \tag{5.13}$$

which can be immediately integrated if ρ is constant. If we integrate (5.13) between z_0 and z, to which the pressure values P_0 and P correspond

$$\int_{P_0}^{P} dP = -g\rho \int_{z_0}^{z} dz \tag{5.14}$$

we obtain

$$P - P_0 = -\rho g\left(z - z_0\right) \tag{5.15}$$

that can be written

$$P = P_0 + \rho gh \tag{5.16}$$

where we have set $h = z_0 - z$. (5.16) is called Stevin's law.

DYNAMICS

Ideal fluids and viscosity

Ideal fluids are :

a) incompressible, that is their density is constant, in time and space;

b) frictionless;

c) homogeneous and isotropic.

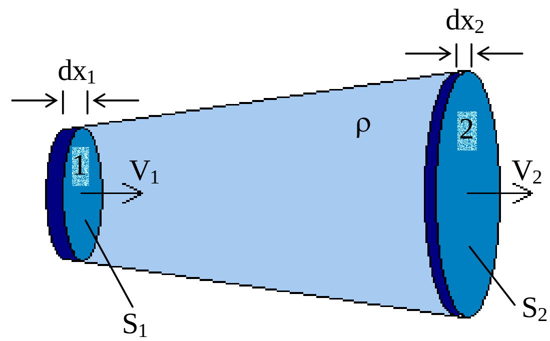
About a): it is clear that liquids are incompressible, but gases are not.

About b): friction ir dependent on viscosity**.** All fluids have a *viscosity*, which describes the amount of friction between nearby regions of the fluid moving at different velocities. Viscosity is what causes *dissipation of energy* in hydrodynamic flow: after you stir your coffee, it eventually stops moving, thanks to frictional losses. The units of viscosity are pressure×time, or mass/

(length×time). The cgs unit of viscosity is 1 g/(cm·sec) = 1 Poise; the SI unit of viscosity is 1 Pa·sec. It is useful to remember that 1 Poise = 0.1 Pa·sec (check this!)

Water, and solutions which are mostly water, have a viscosity close to h_{water} = 0.01 Poise or 0.001 $Pa \cdot s$. This value of viscosity will be used extensively, since the stuff around biomolecules in vivo is mostly water.

The continuity equation



Let us consider a flow tube , in which a fluid of density ρ flows of laminar motion. Be S_1 the section of the tube and v_1 the velocity of the fluid at point 1, and S_2 the section and v_2 the velocity at point 1. The conservation of mass implies that in the unit of time dt the amount of mass of fluid that enters the tube through the section S1, (M_1), be equal to the mass that leaves the tube through S_2 (M_2). At point 1, in the time interval Δt, the fluid travels a distance

$$dx_1 = v_1 dt \qquad (5.17)$$

the volume at point 1 which corresponds to dx_1 is $S_1 dx_1 = S_1 v_1 dt$, and the mass M_1 contained inside this volume is

$$M_1 = \rho S_1 v_1 dt \qquad (5.18)$$

In the same way at point 2 we have

$$M_2 = \rho S_2 v_2 dt \qquad (5.19)$$

The condition that the total mass contained inside the tube must not change can be expressed saying that $M_1 = M_2$; this condition is equivalent to state

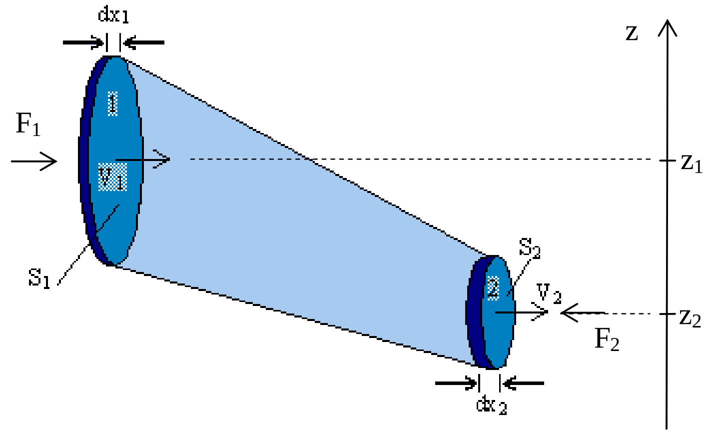
$$\rho S_1 v_1 \Delta t = \rho S_2 v_2 \Delta t \qquad (5.20)$$

which gives:

$$S_1 v_1 = S_2 v_2 \qquad (5.21)$$

(5.21) is known as “equation of continuity”.

Bernoulli's equation



We now derive the fundamental equation of the dynamics of fluids. In the figure a flow tube is represented; on the section S_1 a force F_1 acts, directed in the same direction as the flow. On section S_2 a force F_2 is exerted, directed in the opposite direction (remember that only the pressure acting **on** the fluid is important). We compute in the present case the balance of work and energy (4.19). The masses in the two volumes at the beginning and at the end of the tube are given by (5.18) and (5.19). We have:

$$\Delta K = K_2 - K_1 = \frac{1}{2}\rho S_2 v_2 \Delta t v_2^2 - \frac{1}{2}\rho S_1 v_1 \Delta t v_1^2 \qquad (5.22)$$

The change in the potential energy is

$$\Delta U = U_2 - U_1 = \rho S_2 v_2 \Delta t g z_2 - \rho S_1 v_1 \Delta t g z_1 \qquad (5.23)$$

the last term that must be computed is the work of the pressure forces. The pressures at pooints 1 and 2 are: $P_1 = F_1 / S_1$ and $P_2 = F_2 / S_2$.

The expressions of the work done by F_1 and F_2 are:

$$W_1 = P_1 S_1 v_1 \Delta t \qquad (5.24)$$

where $\Delta x_1 = v_1 \Delta t$. Similarly

$$W_2 = -P_2 S_2 v_2 \Delta t \qquad (5.25)$$

wher the sign depends on the opposite directions of v_2 and F_2. Substituting (5.22), (5.23), (5.24) and (5.25) in (4.19) (with $W_{NC} = W_1 + W_2$) we obtain:

$$P_1 S_1 v_1 \Delta t - P_2 S_2 v_2 \Delta t = \left[\left(\rho S_2 v_2 \Delta t \right) g z_2 - \left(\rho S_1 v_1 \Delta t \right) g z_1 \right] + \left[\frac{1}{2}\left(\rho S_2 v_2 \Delta t \right) v_2^2 - \frac{1}{2}\left(\rho S_1 v_1 \Delta t \right) v_1^2 \right] \qquad (5.26)$$

This equation can be simplified by dividing by Δt and by using the equation of continuity (5.21) obtaining

$$P_1 + \rho g z_1 + \frac{1}{2}\rho v_1^2 = P_2 + \rho g z_2 + \frac{1}{2}\rho v_2^2 \qquad (5.27)$$

which can be written also

$$z_1 + \frac{P_1}{\rho g} + \frac{1}{2}\frac{v_1^2}{g} = z_2 + \frac{P_2}{\rho g} + \frac{1}{2}\frac{v_2^2}{g} \qquad (5.28)$$

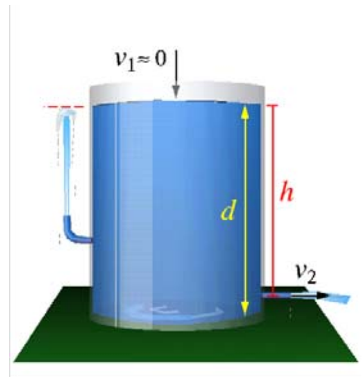
(5.28) is the equation of Bernoulli, and states that the sum of the three terms

$$z+\frac{P}{\rho g}+\frac{1}{2}\frac{v^2}{g}$$

is constant.

Torricelli's law

We now apply the Bernoulli's equation to the case of a fluid flowing from a reservoir, with the radius of the orifice much smaller than that of the reservoir itself. Be r the radius and R the reservoir one; the condition is $R \gg r$; the equation of continuity (5.21) reads:



$$v_1 = v_2\frac{S_2}{S_1} = v_2\frac{r^2}{R^2} \qquad (5.29)$$

and by virtue of the above condition we find that $v_1 \simeq 0$.

The Bernoulli's equation:

$$z_2+\frac{P_2}{\rho g}+\frac{1}{2}\frac{v_2^2}{g} = z_1+\frac{P_1}{\rho g}+\frac{1}{2}\frac{v_1^2}{g} \qquad (5.30)$$

where now $z_1 - z_2 = h$, and $P_1 = P_2$ being both equal to the atmospheric pressure. We then have:

$$\left(z_1 - z_2\right) = h = \frac{1}{2}\frac{v_2^2}{g} \qquad (5.31)$$

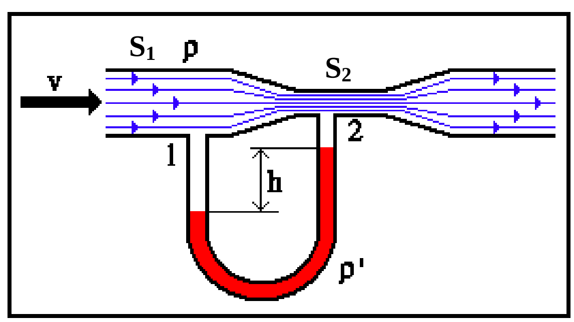
From this equation we obtain:

$$v_2 = \sqrt{2gh} \qquad (5.32)$$

which is called Torricelli's theorem.

The Venturi's tube

An important application of the bernoulli's law is the Tube of Venturi (or simply Venturi). Is consists of a glass pipe that has sections with different diameters and an U-shaped glass tube connecting both sections. One end of the pipe is connected by a rubber hose to a pressurized air supply and the other is open. The U-tube is half filled with colored fluid, whose density ρ' is greater than the density ρ of the fluid that flows in the tube. As a fluid stream flows through the pipe, the different cross-sections will cause different velocities. This results in different pressures being exercised in each water column that is demonstrated by the different red fluid levels.



From the Bernoulli's law (5.28) we have

$$\frac{P_1 - P_2}{\rho \not{g}} = \frac{1}{2\not{g}}\left(v_2^2 - v_1^2\right) \tag{5.33}$$

or

$$2\left(\frac{\Delta P}{\rho}\right) = v_2^2 - v_1^2 \tag{5.34}$$

v_1 and v_2 are linked by the equation of continuity (5.21):

$$v_2 = v_1 \frac{S_1}{S_2} \tag{5.35}$$

so that

$$2\left(\frac{\Delta P}{\rho}\right) = \left(v_1 \frac{S_1}{S_2}\right)^2 - v_1^2 = v_1^2\left(\frac{S_1}{S_2} - 1\right)^2 \tag{5.36}$$

The pressure drop ΔP is related to the different red fluid levels by the relation

$$\Delta P = \left(\rho' - \rho\right) gh \tag{5.37}$$

so that we finally have

$$v_1^2 = \frac{1}{\left(S_1/S_2 - 1\right)}\sqrt{2gh\left(\frac{\rho' - \rho}{\rho}\right)} \tag{5.38}$$

so that the velocity v_1 of the fluid is measured by the different levels in the U-shaped tube.

The Nature of Fluids

The term "fluid" applies to both liquids and gases, and indeed to some things which we think of as solids (ie., glass). The essential differences between fluids and solids can be summarized as follows:

Fluids are shapeless. In more technical terms, we say that fluids do not resist being **"sheared"**. That is, it is relatively hard to bend a solid object, but a fluid "splashes" under the same force.

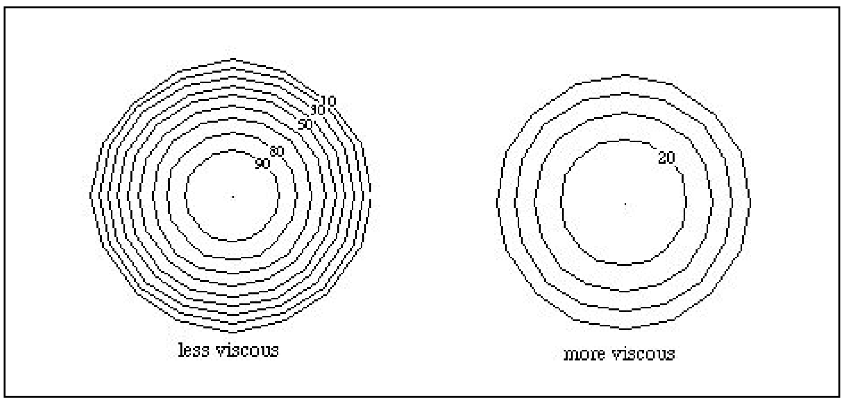
When a force is applied to a fluid, the pressure increases, but whereas the force is directional, the **pressure is omnidirectional**. Consider the force applied to the surface of a glass of water by the atmosphere. That force is downward, but the resulting pressure is felt throughout the water. If we measure the hydrostatic pressure at any given depth, it is the same on all sides of the measuring instrument. In cases where we can ignore hydrostatic pressure and dissipative losses, the omnidirectionality of pressure is equivalent to conservation of energy: the energy density (pressure) is a constant throughout the fluid.

Fluids are "viscous" (but not necessarily viscious!). While fluids do not resist shear, they **do** resist changes in the **rate of change** of shear. The more viscous the fluid, the greater its resistance to changes in the rate of shear: colloquially, we say the fluid is "syrupy". In cgs units, viscosity is measured in Poise: the viscosity of water is .01 Poise (dyne s/cm ^{2} = g / cm s), and the viscosity of blood is .04 Poise.

Fluids flowing past a solid surface obey the "no slip condition". That, is the velocity of the fluid at the solid surface is zero. This may seem surprising, until you recall that your dishwasher often does not clean well when, for instance, the dishes are coated with cheese. The no slip condition means that the water by itself is not as effective as scrubbing with a solid object, and so many of us wipe the dishes before putting them in a dishwasher. Since the velocity of the fluid is nonzero elsewhere, the no slip condition implies that **a "velocity gradient" exists in the flowing fluid**. In a pipe (or blood vessel), the velocity profile across the diameter is essentially parabolic:

$v = \Delta P\,(a^2 - r^2) / 4\,l\,\eta$,

where ΔP is the pressure difference between the "head" (upstream) and "tail" (downstream) ends of the pipe, a is the radius of the pipe, r is the position along the diameter from the center, l is the length of the pipe and η is the viscosity of the fluid. For a cross section of the pipe, the velocity gradient has circular symmetry:



The fact that the velocity is constant on each circle leads us to think of the fluid as flowing in concentric "sheets", but in fact the velocity is a smooth function of distance from the center. For higher viscosity, the velocity gradient is shallower. Essentially, **viscosity opposes the existence of steep velocity gradients**. The larger the viscosity, the "gentler" is the shape of the parabolic velocity gradient, and the velocity is more nearly constant across the section.

Fluids are subject to turbulence. Smooth flow is called "laminar", implying that layers of fluid flow past each other in a sheet-like fashion. The flow is turbulent when eddies and vortices occur. Fluids lose energy through dissipation in both types of flow. During laminar flow, the viscosity causes "frictive" losses (due to "friction") between the sheets of fluid: the fluid resists changes in the velocity gradient, and that costs energy. In turbulent flow, energy is required to create the eddies and vortices, resulting in an energy loss by the fluid as a whole.

Poiseuille's equation

For laminar, non-pulsatile fluid flow through a uniform straight pipe, the flow rate (volume per unit time) is given by Poiseuille's Equation:

$$F = \frac{\Delta P \pi r^4}{8\eta l} / \qquad (5.39)$$

APPENDIX 1

Derivation of (3.28)

From (3.27):

$$mg(1\text{-}V_S\rho)\text{-}bv = m\frac{dv}{dt} \tag{A.40}$$

we find

$$\frac{mdv}{mg(1\text{-}V_S\rho)\text{-}bv} = dt \tag{A.41}$$

with simple manipulations:

$$\frac{dv}{mg(1\text{-}V_S\rho)\text{-}bv}\left(\frac{-b}{-b}\right) = \frac{1}{m}dt \tag{A.42}$$

we obtain

$$\frac{-bdv}{mg(1\text{-}V_S\rho)\text{-}bv} = -\frac{b}{m}dt \tag{A.43}$$

Now we proceed to integrate (A.43)

$$\int\frac{-bdv}{mg(1\text{-}V_S\rho)\text{-}bv} = \int\left(-\frac{b}{m}\right)dt \tag{A.44}$$

and we obtain

$$\ln\left[mg(1\text{-}V_S\rho)\text{-}bv\right] = -\frac{b}{m}t + \mathrm{A} \tag{A.45}$$

$$mg(1\text{-}V_S\rho)\text{-}bv = e^{-\frac{b}{m}t+\mathrm{A}} = Be^{-\frac{b}{m}t} \tag{A.46}$$

We can set the initial condition so that the particle is at rest at t=0, i.e.:

$$t = 0 \Leftrightarrow v(0) = 0$$

These values, inserted in (A.46), give

$$B = mg\left(1 - V_S\rho\right) \tag{A.47}$$

By substituting (A.47) in (A.46), with simple algebraic calculations, we finally obtain:

$$v = \frac{mg}{b}\left(1 - V_S\rho\right)\left(1 - e^{-\frac{b}{m}t}\right) \tag{A.48}$$